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DP: Damped Driven Chaotic Pendulum

Abstract

Over the course of three experiments, the behaviour of a torsion pendulum is analysed. In Experiment 1, the eddy current brake is used to demonstrate an underdamped, critically damped, and overdamped system (impossible due to current limits). In Experiment 2, frequency is plotted against applied voltage for calibration, the resonance frequency of the system is found, the Quality Factor is determined, and the brake is turned on to demonstrate a damped driven system. In Experiment 3, the double-well potential, period-2 oscillations, and chaotic motion are analysed.

Introduction

In this experiment, a torsion pendulum's behaviour is analysed according to damping (by an eddy current brake) and driving (by a motor) via phase portraits, theta vs t plots, and frequency spectra. When two unequal weights are placed on either side of the indicator (set at 0 degrees at equilibrium), the pendulum gains two equilibrium points and oscillates between them. As voltage is incrementally decreased, the pendulum's plot of angle vs time (theta vs t) strays from periodicity, exhibiting what is defined as chaotic behaviour.

The equation of motion for this pendulum is given as Equation 1 below (from reference [1]) such that θ is the angular position, I is the moment of inertia, A is the amplitude of the torque, $Acos(\omega_D t)$ is the driving torque term, $\gamma = \frac{\lambda}{I}$ and $\omega_0 = \sqrt{\frac{k}{I}}$.

$$\frac{d^2\theta}{dt^2} + \gamma\left(\frac{d\theta}{dt}\right) + \omega_0^2\theta = \frac{A}{I}\cos(\omega_D t)$$

Equation 1: Motion of a Torsion Pendulum

For a system with damping but no driving force, Equation 2 below applies (from reference [1]). This equation can be modified for small, large, and critically damped systems (relevant modifications in Analysis where required).

$$\theta = C_1 \exp(-\left(\frac{\gamma}{2} - \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}\right)t) + C_2 \exp(-\left(\frac{\gamma}{2} - \sqrt{\left(\frac{\gamma}{2}\right)^2 - \omega_0^2}\right)t)$$

Equation 2: Motion with no Driving Force

For a damped and driven system, Equation 3 applies (from reference [1]). This equation is valid for small damping for a damped driven torsion pendulum. The term with the 'a' coefficient represents the transient behaviour, and as such disappears with time.

$$\theta = ae^{-\frac{\gamma}{2}t}\cos(\omega_0 t - \Phi) + b\cos(\omega_D t + \delta)$$

Equation 3: Lightly Damped Driven Motion

Resonance occurs when the amplitude of the theta vs t plot is at its maximum. This is because the frequency of the pendulum is equal to the frequency of the motor, and thus maximum displacement from equilibrium is achieved. In this experiment, this is accomplished by using the voltage vs frequency plots obtained from experimental data and determining the resonant frequency according to the voltage at which resonance occurs.

Experimental Details

The experimental setup included the sensor-CASSY and corresponding CASSY Lab program on the computer used to obtain experimental data before transferring it to an Origin file for analysis. The torsion pendulum or "Pohl's wheel" featured an indicator arrow which was vertical (0 degrees) at equilibrium. The counterweight (in the form of a piece of Blu-tack) had to be a weight such that it would allow for motion of the pendulum without resulting in torque or prevent accurate motion detection. The eddy current brake was used for damping and the driving motor was used to drive the system. The motion transducer box contained a small wheel that acted as a second track for the string carrying the counterweight.

Experimental Procedure:

The CASSY Lab program was opened, and from the Settings window, **General>COM1 (drop down menu)>CASSY** was configured to generate data from the sensor-CASSY for measurements. From the CASSY tab, the top box (BMW-box) was selected. The file PendulumSample was loaded such that phase portraits, frequency spectra, and theta vs t plots would be generated for a given pendulum measurement.

Experiment 1:

The pendulum was displaced from equilibrium and measurements were taken with no damping or driving force to generate the required plots. The theta vs t plot maximum amplitudes were plotted such that an exponential fit could be generated to determine the intrinsic damping coefficient.

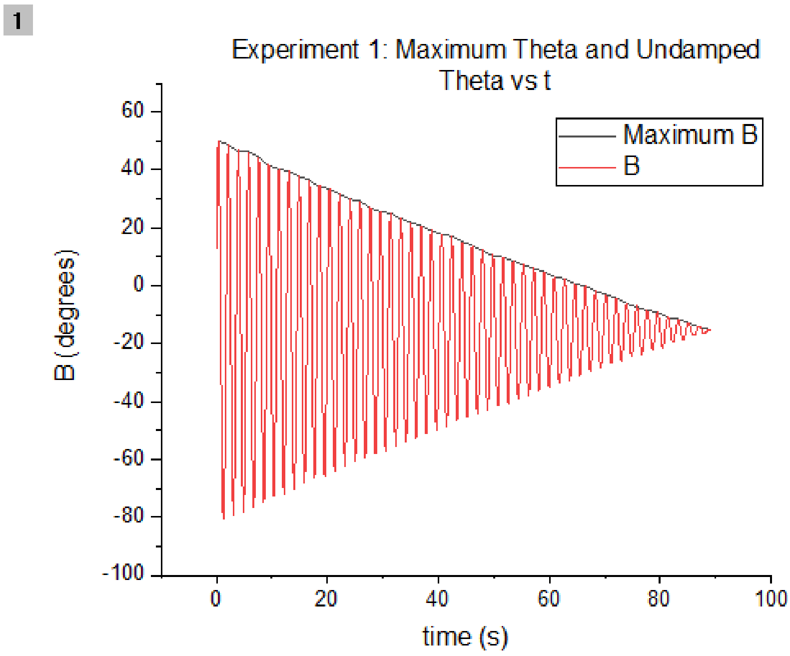

Figure 2: Experiment 1 Maximum Theta and Undamped Theta vs t

(On same plot for comparison/understanding of the derivation of the intrinsic damping coefficient)

The eddy current brake was then turned on and current increased with a noted maximum of 1.5 A to obtain critically damped and overdamped plots.

Experiment 2:

1) By increasing the voltage from 2 to 12 V and measuring the frequency at every 2 V interval, a plot of voltage vs frequency was generated.

2) By slowly increasing the voltage from 2 to 12 V, resonance was found by noting the voltage at which the amplitude of the theta vs t plot was at its maximum (this was then converted to frequency using the fitted voltage vs frequency plot from 1)). A plot of amplitude vs frequency was then generated.

3) The quality factor was then determined by generating a Lorentzian fit to the resonance peak and noting the resonant angular frequency and damping coefficient (noting the frequency at which resonance occurs).

4) A damped driven system was achieved by turning on the eddy current brake along with the motor. A frequency voltage calibration curve (frequency vs voltage plot as in part 2)), resonance curve (theta vs t), and constants were calculated (resonant angular frequency, damping coefficient, and quality factor) and compared to the values/plots from part 2). Phase lag was then observed by slowly increasing frequency.

Experiment 3:

Two weights of different masses were added to either side of the indicator on the torsion pendulum at positions such that the pendulum would be able to oscillate in two different equilibrium points and move between them without tampering. This allowed for the analysis of a double-well system.

The current was set to 0.6 A to prepare for period-2 oscillations. The voltage was set far above resonance such that the pendulum was only able to oscillate around a single equilibrium point. A phase portrait was generated, noting an "egg" resemblance in the plot.

From the high voltage above resonance, voltage was decreased by 0.2 V until resonance was reached to determine the pendulum's period. Phase portraits, frequency spectra, and theta vs t plots were generated.

From the conditions in the period-2 oscillation analysis, the voltage was decreased by 0.1 V until its behaviour was no longer periodic. Phase portraits, frequency spectra, and theta vs t plots were generated for this analysis of "chaos".

Results and Analysis

Experiment 1

Undamped/Lightly damped:

The first plot generated was the phase portrait for an undamped system (no manual damping, but friction is still present), this plot is Figure 3 below.

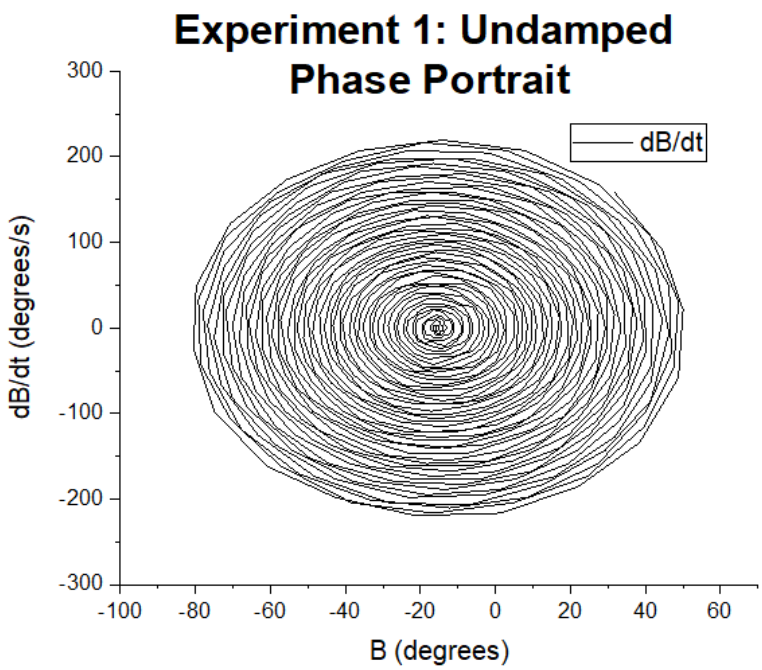

Figure 3: Experiment 1 Undamped Phase Portrait

The second plot for Experiment 1 was the theta vs t graph for an undamped system (Figure 4). The maximum amplitude points were obtained and plotted as a separate line graph in order to obtain an exponential fit to determine the intrinsic damping coefficient. One may note that the decay of this plot mirrors the expectation sketch in the lab notebook for a lightly damped system.

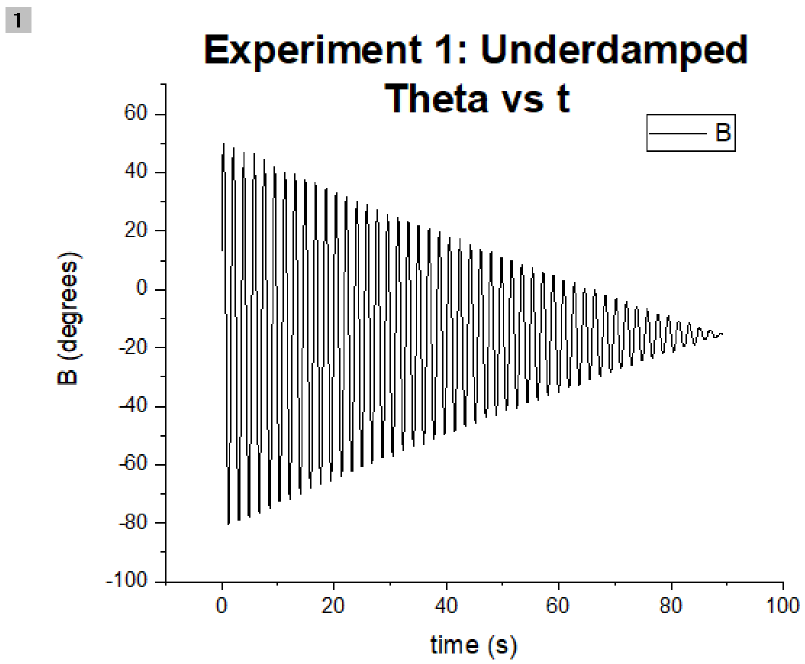

Figure 4: Experiment 1 Underdamped Theta vs t

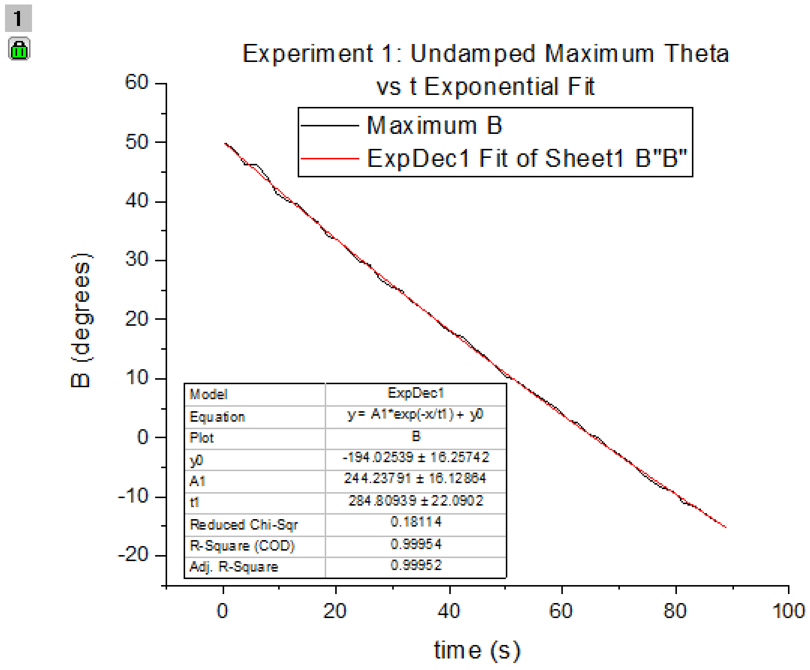

Figure 5: Experiment 1 Undamped Maximum Theta vs t Exponential Fit

After the exponential fit was generated, the equation of motion for a lightly damped system from the experimental script for this experiment on Blackboard was referenced. This equation is an alteration of Equation 2 (in the Introduction) and is given below as Equation 4 (from reference [1]).

$$\theta = aexp\left(-\frac{\gamma}{2}t\right)\cos(\omega t + \Phi)$$

Equation 4: Motion of a Pendulum with no Manual Damping

The intrinsic damping coefficient may then be found by setting the terms in the exponential from the fit and Equation 4 equal to one another and solving for γ (the variables 't' and 'x' can be cancelled since they both represent the values of the horizontal axes).

$$-\frac{\gamma}{2}t = -\frac{x}{t1} \equiv \frac{\gamma}{2} = \frac{1}{t1} \rightarrow \gamma = \gamma_0 = \frac{2}{t1} = \frac{2}{285} \cong 0.007$$

Critically damped:

The phase portrait, frequency spectrum, and theta vs t plots for a critically damped system are given by Figures 6, 7, and 8 respectively. One may note that Figure 8 resembles the expectation sketch from the lab notebook for this experiment with the exception of the placement of the negative quadratic resembling section occurring after as opposed to before t=0.

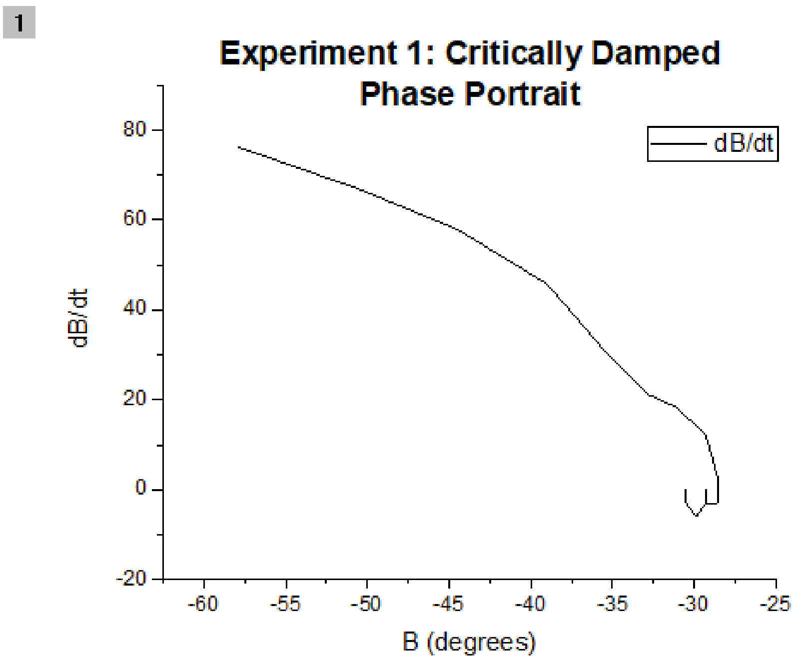

Figure 6: Experiment 1 Critically Damped Phase Portrait

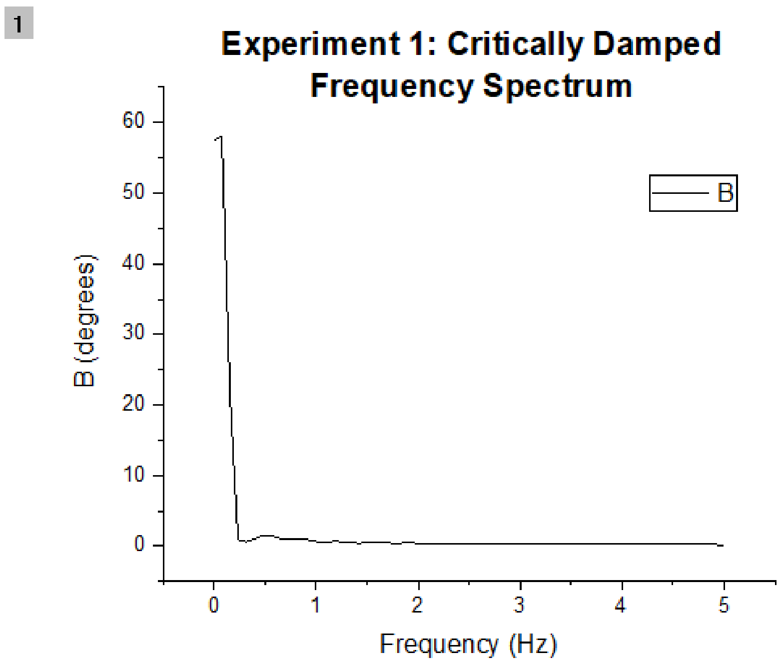

Figure 7: Experiment 1 Critically Damped Frequency Spectrum

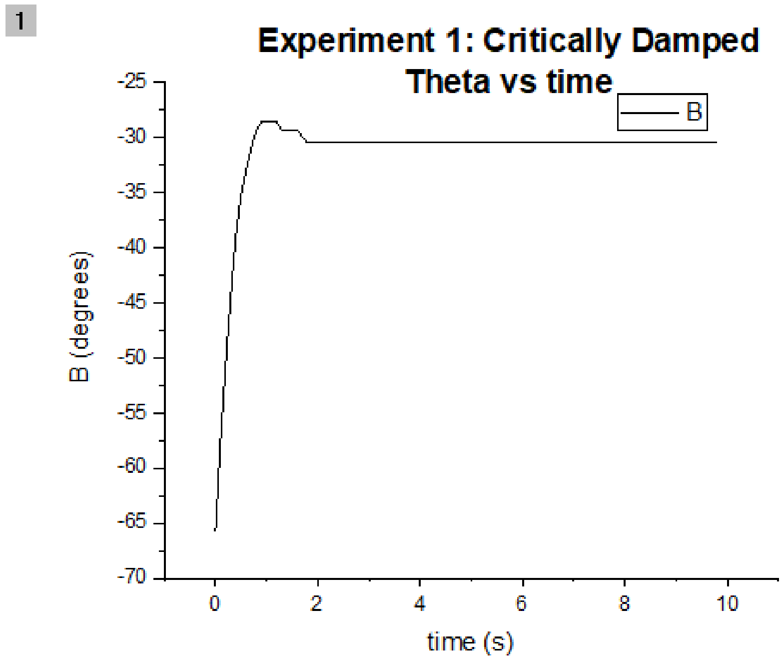

Figure 8: Experiment 1 Critically Damped Theta vs time

Overdamping could not be achieved since the experimental script imposed a limit of 1.5 A for the current setting and critical damping occurred around the current limit. A screenshot of the expectation from the lab notebook sketches is shown as Figure 9 below.

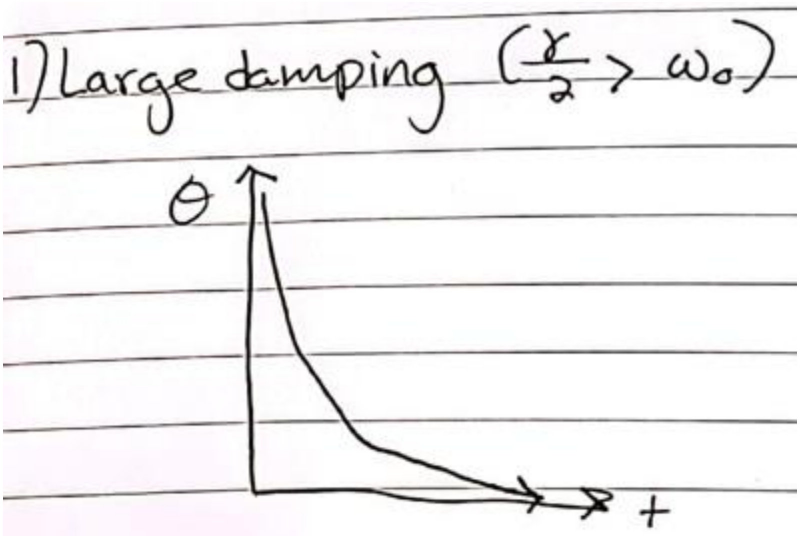

Figure 9: Overdamping Theta vs t Expectation from Lab Notebook

Experiment 2:

1) Frequency calibration

The transient motion phase portrait (Figure 10) of the pendulum before frequency calibration was obtained.

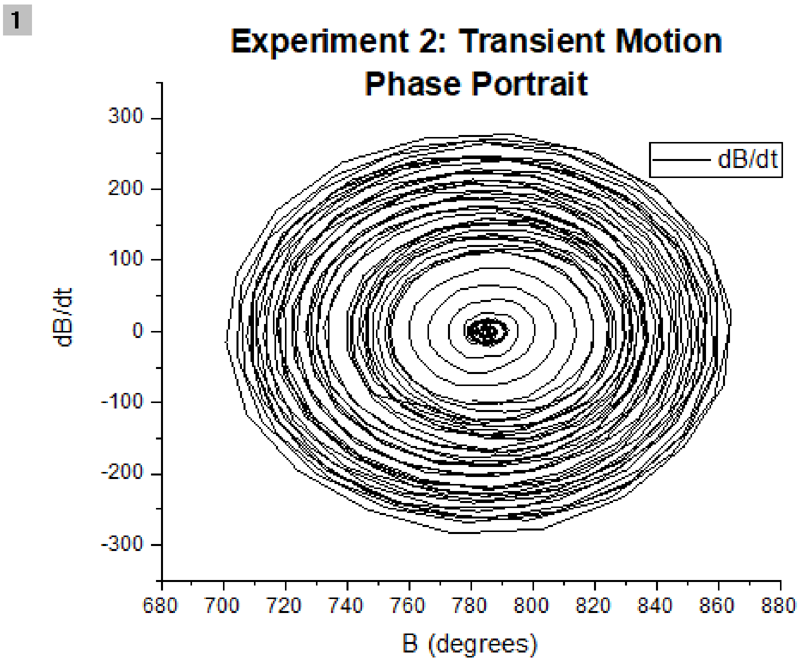

Figure 10: Experiment 2 Transient Motion Phase Portrait

Frequency calibration was achieved by fitting a linear function to the frequency vs voltage plot (Figure 11 below). With this information, frequency values for a given voltage could be determined.

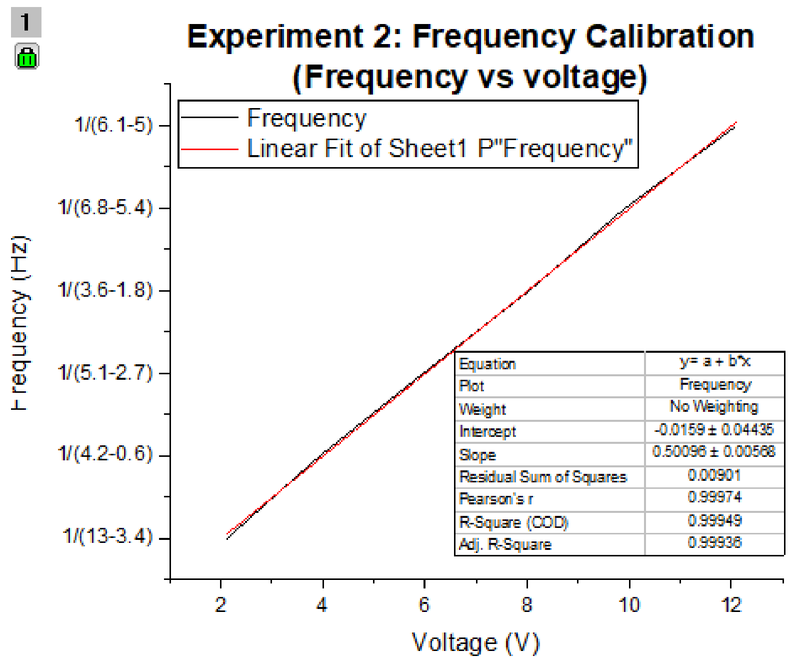

Figure 11: Experiment 2 Frequency Calibration (Frequency vs voltage) Linear Fit

2) Resonance

Resonance was found at approximately 7.95 V. Figure 12 shows the plot obtained of amplitude vs frequency around resonance (where frequency was determined from voltage according to the relationship between frequency and voltage found from Figure 11).

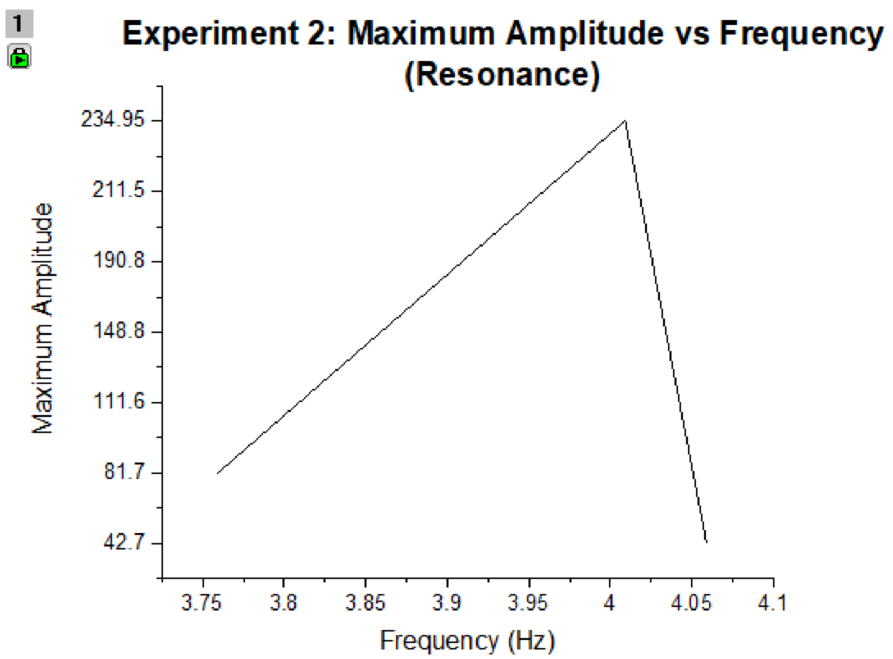

Figure 12: Experiment 2 Maximum Amplitude vs Frequency (Resonance)

3) Quality Factor

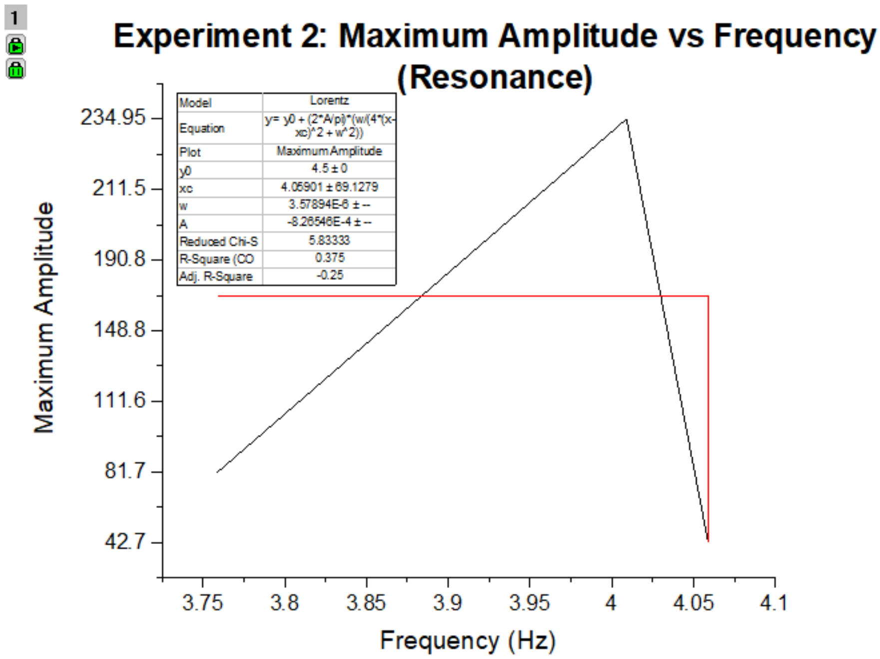

Figure 13: Experiment 2 Maximum Amplitude vs Frequency Lorentzian Fit

One may note that the fit in Figure 13 does not look like the example shown in the experimental script (Figure 14 below), and that in a plot of maximum angle vs frequency, there should be a peak at approximately the resonant frequency. After calculations, this is shown to be true.

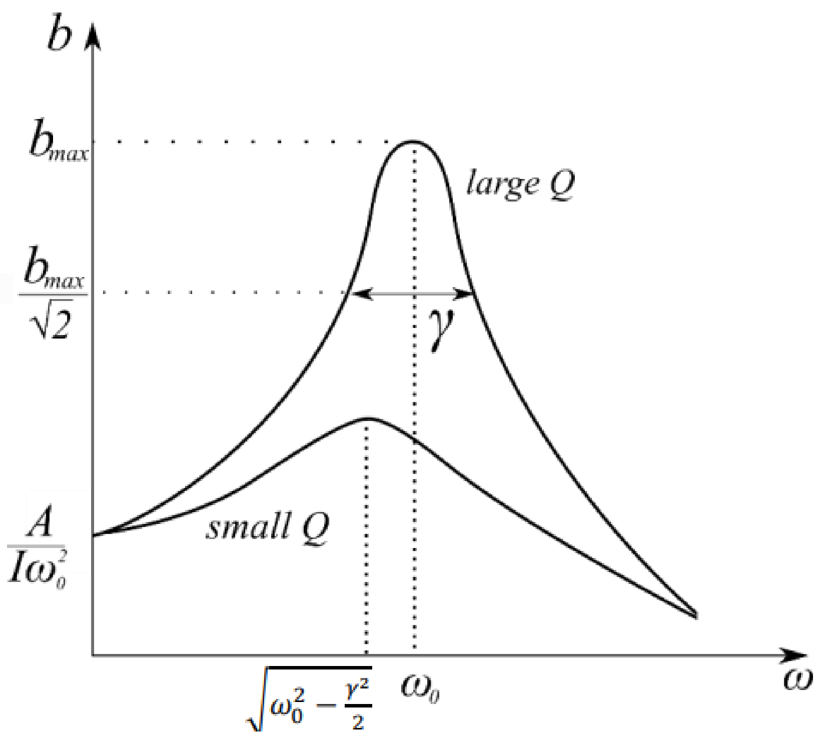

Figure 14: Lorentzian Function from Experimental Script

This error in fit function is due to the fact that not enough points were used in the frequency vs maximum amplitude plot, which will likely impact the validity of the obtained results. However, since the script states that $\omega_0 = 2\pi f_0$, Figure 14 shows that ω_0 is the frequency of maximum amplitude, and f_0 is the resonant frequency (determined to be 7.95 V which is 3.96 Hz when converting using the frequency to voltage relationship), $\omega_0 = 2\pi f_0 = 2\pi(3.96\ Hz) = 24.9\ Hz$. Since γ_0 is the FWHM, the fit from Origin must be consulted, noting the error which will likely arise from the result. Since the resonant peak varies so much in the range considered, one may expect a large Q factor (narrow resonance peak).

Book2 *

Notes
Input Data
Parameters

		Value	Standard Error	t-Value	Prob>|t|	Dependency
Maximum Amplitude	y0	4.5	2.01383	2.23455	0.11152	0.79378
	xc	4.059	57.99721	0.06999	0.94861	0.99161
	w	0.0057	6.93236	8.22121E-4	0.9994	1
	A	-0.03132	38.10824	-8.21799E-4	0.9994	1
	H	-3.49823	3.14798			

Figure 15: Origin Lorentzian Fit Parameters

One can determine from Figure 15 that the expected peak (xc) value from the Lorentzian fit in origin is ~4.059 which is approximately equal to 3.96 Hz (calculated value). The FWHM (w) is estimated as 0.0057~0.006. Compared to the damping coefficient from Experiment 1 (0.007 Hz), the damping coefficient for this system is approximately 0.001 Hz less. Therefore, the Q factor (noting error due to experimental inaccuracy) may be calculated as 4150 using the formula given in the experimental script (dimensionless, so both values must have units of Hz).

$$Q = \frac{\omega_0}{\gamma_0} = \frac{24.9\ Hz}{0.006\ Hz} = 4150$$

4) Damped driven:

As in part 1), the frequency vs voltage was plotted (Figure 14) to allow for conversion between the values.

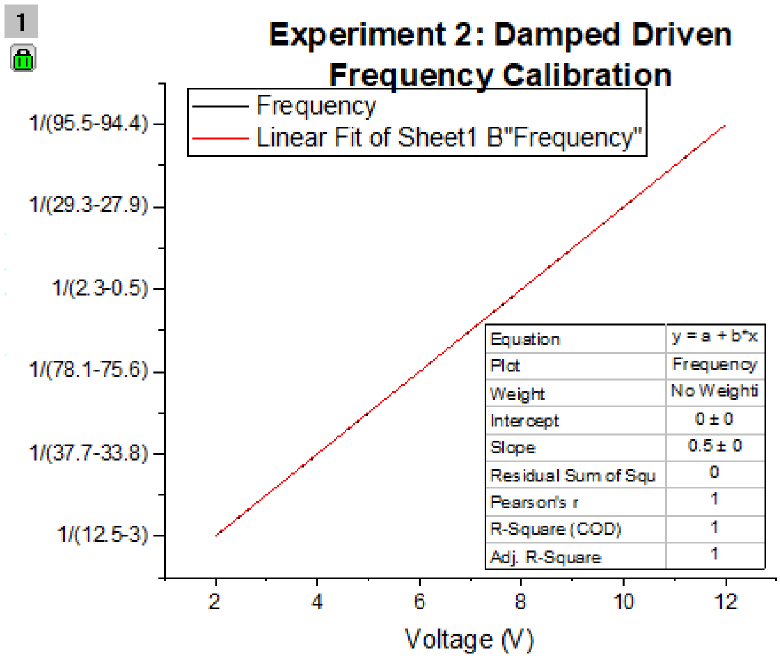

Figure 14: Experiment 2 Damped Driven Frequency Calibration

Figure 15 shows the resonance curves of the undamped and damped driven pendulums (space between the two lines is due to the measurements slowly moving up due to a string which caught and displaced the apparatus).

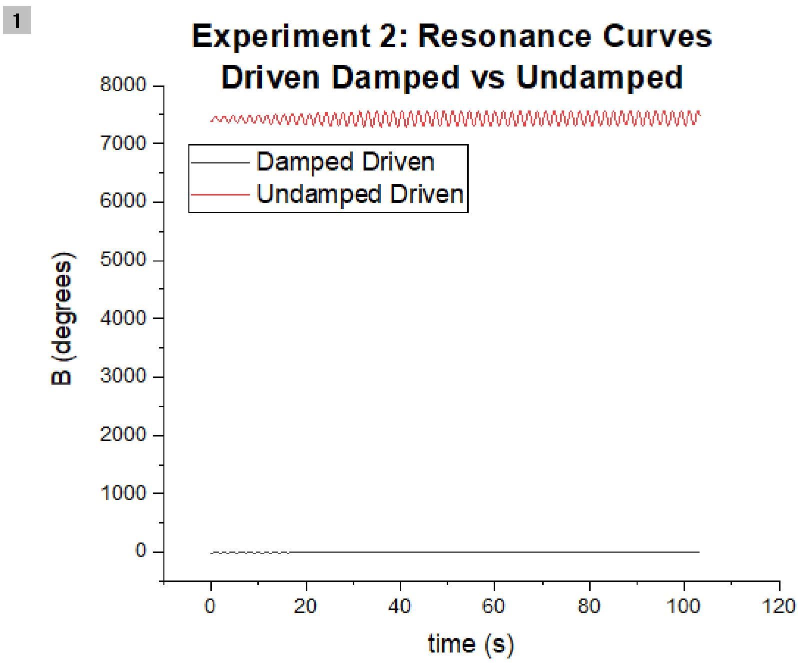

Figure 15: Experiment 2 Resonance Curves Driven Damped vs Undamped

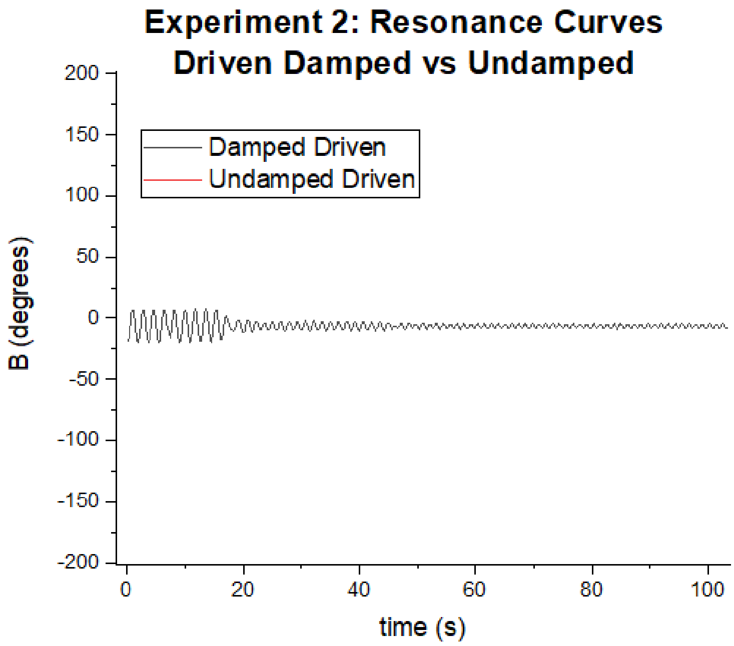

Figure 16: Figure 15 Zoomed in to Show Only Damped Driven Resonance Curve

Due to an error in value transcription, the frequencies/voltages obtained for this portion were not recorded correctly, and thus the Q-factor and damping coefficient could not be calculated. However, the resonant voltage was noted as 7.67 V, which can be converted to frequency using the voltage-frequency relation and thus converted to a value for ω_0.

$$\omega_0 = 2\pi f_0 = 2\pi\left(0.5\ \frac{Hz}{V} * 7.67\ V\right) = 24.1\ Hz$$

Compared to the ω_0 value for an undamped driven pendulum (determined to be 24.9 in part 2)), the value for a damped driven pendulum is slightly lower, which is logically sound since the ω_0 value is directly dependent on the frequency, and when weight or damping is added, the pendulum is expected to move at a lower frequency.

<u>Experiment 3:</u>

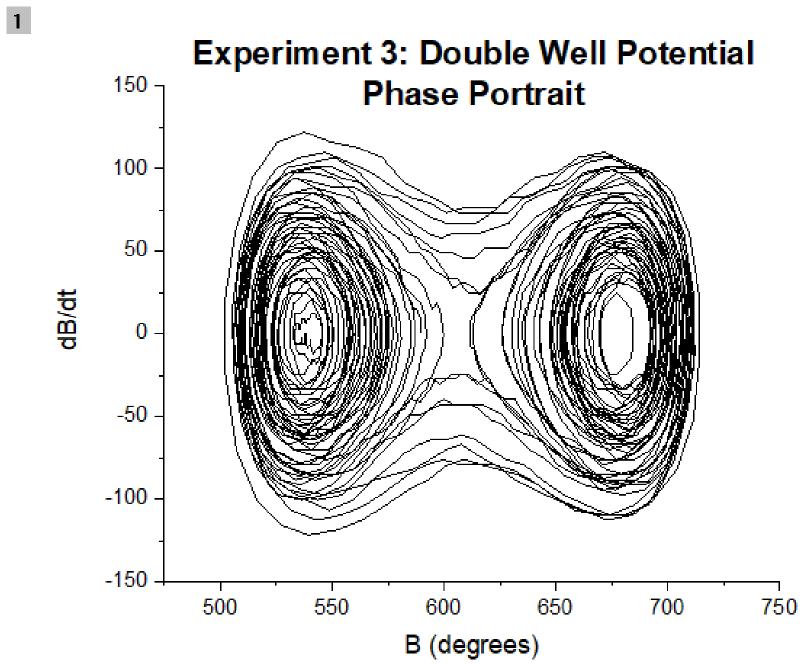

Figure 17: Experiment 3 Double Well Potential Phase Portrait

Figure 17 demonstrates the phase portrait of a pendulum with weights of unequal masses (indicated as equal in the experimental script, but the desired effect was not achieved with equal masses, so unequal masses were attempted and ultimately successful) such that it possesses two equilibrium points (demonstrated by the spiral centres of the plots) and can oscillate between them (demonstrated by the lines connecting the two spirals).

Figures 18 and 19 demonstrate the period-2 oscillation phase diagram and theta vs t plot respectively. The frequency spectrum was generated but not saved due to an error arising from time management issues.

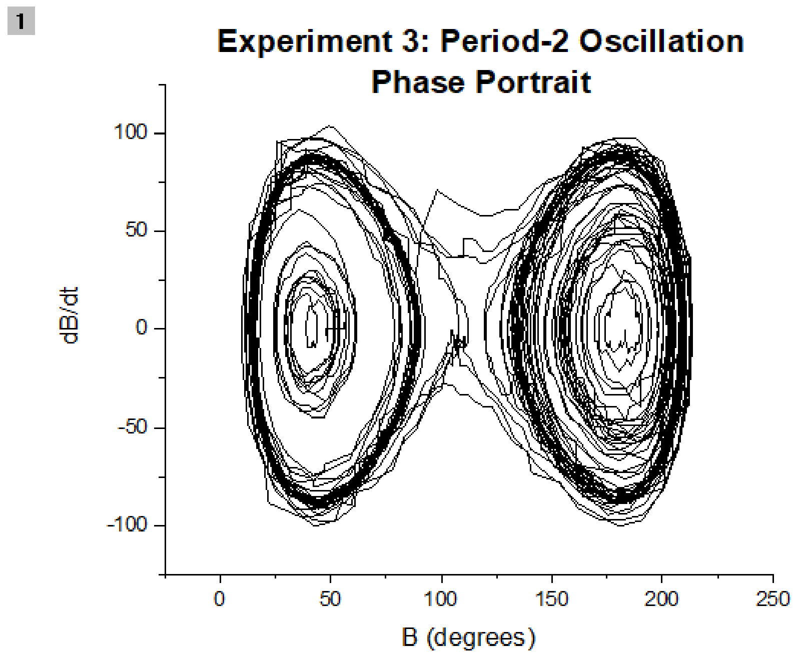

Figure 18: Experiment 3 Period-2 Oscillation Phase Portrait

In Figure 18, one may note that as in Figure 17, there are two spirals with a (more jagged) series of connecting lines between them.

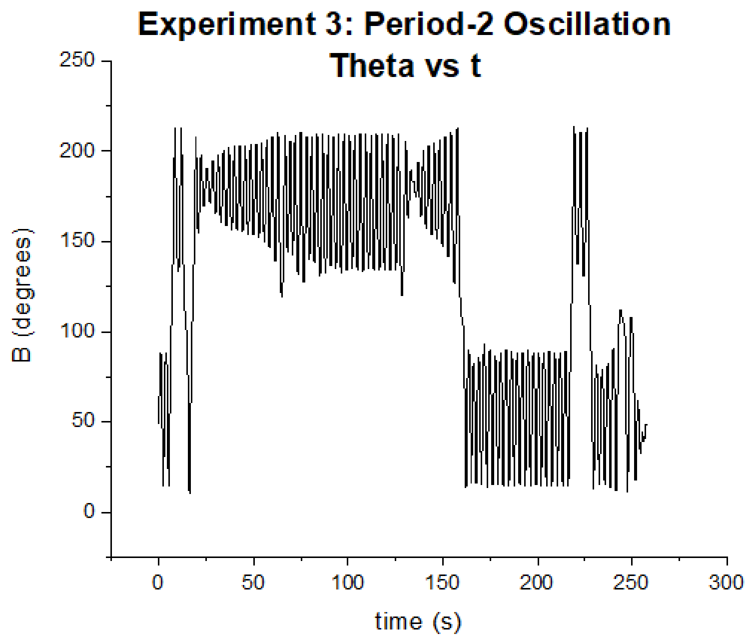

Figure 19: Experiment 3 Period-2 Oscillation Theta vs t

In Figure 19, one can observe that there are intervals of periodic behaviour followed by jumps to approximately the same theta (y-axis) values. The values that the function oscillates between correspond to the two equilibrium positions on the pendulum (which appear to be approximately 175 and 50 degrees) and the 'jump' is the pendulum's motion oscillating from one equilibrium point to the other. This is more clearly represented in Figure 19b below.

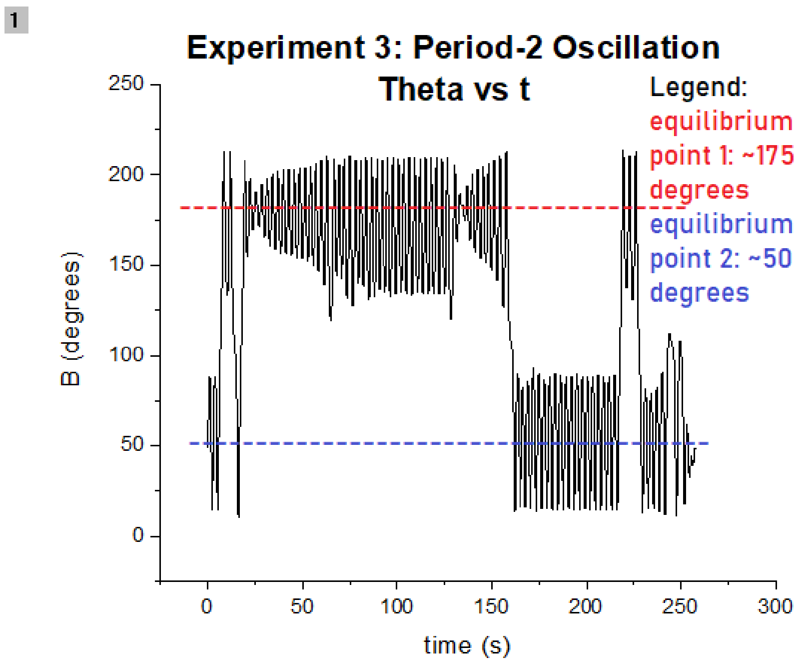

Figure 19b: Experiment 3: Period-2 Oscillation Theta vs t

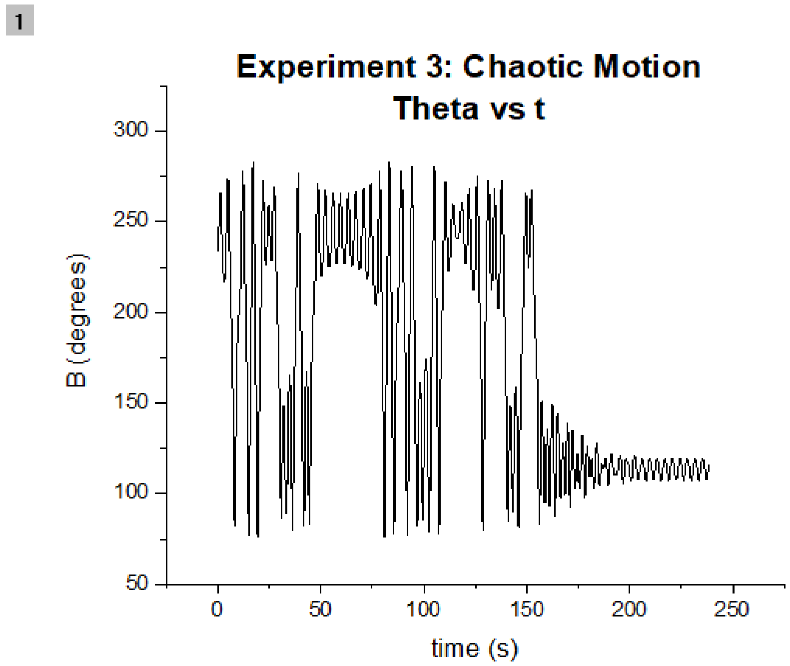

Figure 20: Chaotic Motion Theta vs t

Figure 20 demonstrates the chaotic motion of the torsion pendulum. One may note the erratic behaviour in the interval between 0 and approximately 150 s. Due to the return to periodicity in the interval past 150 seconds, one may assume that enough time was not taken before the measurement to accurately show chaotic behaviour.

Conclusions

Experiment 1: The underdamped and critically damped systems analysed were consistent with the sketched predictions from the lab notebook. The underdamped (no manual damping, only friction) damping coefficient (or the intrinsic damping coefficient) was found to be 0.007 Hz. Overdamping was impossible since the limit of current for the system was 1.5 A at which point critical damping was observed.

Experiment 2:

1) The frequency vs voltage plot was generated for the undamped driven system and used to find frequency values corresponding to a given voltage.

2) The resonance curve was generated for the undamped driven system. A plot of maximum amplitude vs frequency was generated.

3) The quality factor was determined to be 4150 for the undamped driven system although the Lorentzian function fitted to the maximum theta vs frequency plot was visibly inaccurate due to not including enough data points.

4) The quality factor could not be determined for the damped driven system due to an error in data transcription, but the resonant angular frequency was determined to be 24.1 (less than that for an undamped driven system).

Experiment 3:

The phase portrait for the double-well oscillations was generated. The period-2 phase portrait, frequency spectrum, and theta vs t plots were generated, but the frequency spectrum was not saved due to time management issues. The chaos was recorded as a theta vs t graph, but its return to periodicity indicates that not enough time was taken before obtaining the measurement.

References

[1] DP: Damped Driven Chaotic Pendulum Experimental Script on Blackboard